

A remark on a result of Helfgott, Roton and Naslund

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Abstract

Let $F(X) = \prod_{i=1}^k (a_i X + b_i)$ be a polynomial with a_i, b_i being integers. Suppose the discriminant of F is non-zero and F is admissible. Given any natural number N , let $S(F, N)$ denotes those integers less than or equal to N such that $F(n)$ has no prime factors less than or equal to $N^{1/(4k+1)}$. Let L be a translation invariant linear equation in 3 variables. Then any $A \subset S(F, N)$ with $\delta_F(N) := \frac{\text{card}(A)}{\text{card}(S(F, N))} \gg_{\epsilon, F, L} \frac{1}{(\log \log N)^{1-\epsilon}}$ contains a non-trivial solution of L provided N is sufficiently large.

Given $A \subset \mathbb{N}$ and a natural number N we set $A(N) := A \cap [1, N]$. Given any natural number k we write $\log_k N := \underbrace{\log \dots \log N}_{k\text{-times}}$. Given a subset P_1 of the set of primes P ,

we define the relative density $\delta_P(N) = \frac{\text{card}(P_1(N))}{\text{card}(P(N))}$. In [4], Ben Green showed that any subset P_1 of the set of primes P , with relative density $\delta_P(N) \geq c(\log_5 N / \log_3 N)^{1/2}$ for some $N \geq N_0$, where c and N_0 are absolute constants, contains a non-trivial 3-term arithmetic progression. In [5], H. Helfgott and A. De. Roton improved this result to show that the same conclusion holds under the weaker assumption that the relative density $\delta_P(N) \geq c(\log_3 N / \log_2 N)^{1/3}$. In [6], Eric Naslund, using a modification of the arguments of Helfgott and Roton, showed that the result holds under even a weaker assumption $\delta_P(N) \geq c(\epsilon)(1/\log_2 N)^{1-\epsilon}$, where $\epsilon > 0$ is any real number and $c(\epsilon) > 0$ is a constant depending only on ϵ . The purpose of this note is to observe that the arguments of Helfgott and Roton [5] and Eric Naslund [6] gives a more general result, namely Theorem 0.1 stated below.

Let $F(X) = \prod_{i=1}^k (a_i X + b_i) \in \mathbb{Z}[X]$ with $a_i \in \mathbb{Z} \setminus \{0\}$ and $b_i \in \mathbb{Z}$. Moreover we suppose that

- (i) the discriminant of F ; $\Delta(F) = \prod_{i=1}^k a_i \prod_{i \neq j} (a_i b_j - b_i a_j) \neq 0$ and
- (ii) is admissible that is to say that for all primes p , there exists $n \in \mathbb{Z}$ such that $F(n) \not\equiv 0 \pmod{p}$.

Hardy-Littlewood conjecture predicts that for any F as above, the number of integers $n \leq N$ such that $a_i n + b_i$ is prime for all i is asymptotically equal to $c(F) \frac{N}{\log^k N}$. This is not known except the case when F is a linear polynomial. However using Brun's sieve we know a lower bound for the number of $n \leq N$ such that the number of prime factors of $a_i n + b_i$ is at most $4k + 1$. Given any real number $z > 0$, let $P(z) = \prod_{p \leq z} p$ and we set

$$S_F(N, z) = \{n \leq N : \gcd(F(n), P(z)) = 1\}. \quad (1)$$

Then using Brun's sieve [2, see page number 78, (6.107)], we know the following lower bound

$$\text{Card} \left(S_F(N, N^{1/(4k+1)}) \right) \geq c_1(F) \frac{N}{\log^k N}, \quad (2)$$

where $c_1(F) > 0$ is a constant depending only upon F . Given any $A \subset S_F(N, N^{1/(4k+1)})$ we define the *relative density* $\delta_F(N)$ of A to be

$$\delta_F(N) = \frac{\text{Card}(A(N))}{\text{Card} \left(S_F(N, N^{1/(4k+1)}) \right)}. \quad (3)$$

For the brevity of notation, we shall also write $\delta(N)$ or simply δ to denote $\delta_F(N)$.

Let $s \geq 3$ be a natural number and

$$L := c_1 x_1 + \cdots + c_s x_s = 0 \quad (4)$$

be a linear equation with $c_i \in \mathbb{Z} \setminus \{0\}$. The linear equation L is said to be *translation invariant* if $\sum_i c_i = 0$. A solution (x_1, \dots, x_s) of L is said to be *non-trivial* if for some i, j we have $x_i \neq x_j$.

Theorem 0.1. *Let k be a natural number, $F(X) = \prod_{i=1}^k (a_i X + b_i) \in \mathbb{Z}[X]$ with $a_i \in \mathbb{Z} \setminus \{0\}$ and $b_i \in \mathbb{Z}$ for all i . Suppose that F is admissible and the discriminant of F is non-zero. Let L be a translation invariant linear equation as defined in (4) with $s = 3$. Then given any $\epsilon > 0$ there exists a constant $c(F, L, \epsilon) > 0$ and a natural number $N(F, L, \epsilon)$ such that the following holds. Given any $N \geq N(F, L, \epsilon)$, any set $A \subset S_F(N, N^{1/(4k+1)})$, with*

$$\delta_F(N) \geq c(F, L, \epsilon) \frac{1}{(\log \log N)^{1-\epsilon}}, \quad (5)$$

contains a non-trivial solution of L .

Remark 0.2. (i) Let P be the set of primes. In the above theorem, taking $F(n) = n$, $L := x_1 + x_2 - 2x_3 = 0$, and A to be a subset of primes $P(N)$, with $\text{Card}(A) \geq \delta \text{Card}(P(N))$, with δ satisfying (5), one recovers the result of Eric Naslund [6] stated above.

(ii) We say that a prime p is a Chen prime if $p + 2$ has at most two prime factors and also any prime factor of $p + 2$ is greater than $p^{1/10}$. Green and Tao [3, Theorem 1.2] had shown that Chen primes contain a non trivial 3-term arithmetic progression. In the above theorem taking $F(X) = X(X + 2)$, $L := x_1 + x_2 - 2x_3 = 0$, we obtain any subset A of Chen primes with relative density δ , with δ satisfying (5), contains a non-trivial 3-term arithmetic progression.

Definition 0.3. Let L be a translation invariant linear equation in s variables as in (4).

(i) Let $h_L : (0, 1) \rightarrow \mathbb{R}$ be a non-negative function satisfying the following. Given any prime P and a set $A \subset \mathbb{Z}/P\mathbb{Z}$ with $\text{Card}(A) \geq \eta P$, the number of solutions of L in A is at-least $h_L(\eta) P^{s-1}$.

(ii) Let $g_L : \mathbb{N} \rightarrow \mathbb{R}_0^+$ be a monotonically decreasing function with $\lim_{N \rightarrow \infty} g_L(N) = 0$ and satisfying the following properties. There exists a natural number N_g such that given any $N \geq N_g$, any set $A \subset [1, N]$ with $|A| \geq g_L(N)N$ contains a non-trivial solution of L . Given $\eta > 0$, let $g_L^{*-1}(\eta)$ denotes the smallest natural number m such that $g_L(m) \leq \eta$.

Remark 0.4. (i) Let g_L be a function as in Definition 0.3. When the number of variables s in L is equal to 3, then using an arguments due to Varnavides, it can be shown that the function $h_L(\eta) = \frac{\eta}{(2g_L^{*-1}(\eta/2))^2}$ is a function satisfying the properties as in Definition 0.3 (i). If we can find a similar relation between the function g_L^* and h_L for $s > 3$, then the result of Theorem 0.1 can be extended for $s > 3$ using the following result of Thomas Bloom.

(ii) In [1], Thomas Bloom showed that there exists an absolute constant $c > 0$ depending only on L such that the function $g_L(N) = c \left(\frac{\log^5 \log N}{\log N} \right)^{s-2}$ satisfies the above properties. In this case $g_L^{*-1}(\eta) \leq \exp(c_1 \eta^{-1/(s-2)} \log^6 \log(\frac{1}{\eta}))$ with $c_1 > 0$ being a constant depending only upon L .

Let $z = \frac{\log N}{3}$ and $M = \prod_{p \leq z} p$. For any $b \in \{0, 1, \dots, M-1\}$, we set

$$A_b = \{n : n \in A, n \equiv b \pmod{M}\}$$

We notice that

$$A_b \subset \{n \leq N/M : \gcd(F(b + nM), P(N^{1/(4k+1)})) = 1\}.$$

The following lemma is an easy consequence of W -trick due to Ben Green.

Lemma 0.5 (W-trick). *There exists a $b_0 \in \{0, 1, \dots, M-1\}$ such that $\gcd(F(b_0), M) = 1$ and*

$$\text{Card}(A_{b_0}) \geq c(F) \frac{\delta \log^k \log N}{\log^k N} \frac{N}{M},$$

where $c(F) > 0$ is a constant depending only upon F and δ is as in (3).

Proof. Since $z \leq N^{1/(4k+1)}$, it follows that if $A_b \neq \emptyset$, then $F(b) \not\equiv 0 \pmod{p}$ for all $p \leq z$. Now for p which does not divide $\Delta(F) \prod_{i=1}^k a_i$, the number of solutions $n \in \mathbb{Z}/p\mathbb{Z}$ of the equation $F(n) \equiv 0 \pmod{p}$ is equal to k . Let $\Delta'(F) = \Delta(F) \prod_{i=1}^k a_i$. Then using Chinese remainder theorem, it follows that the number of $b \in \{0, 1, \dots, M-1\}$ such that A_b is not an empty set is at most $\frac{\prod_{p \leq z} (p-k) \prod_{p | \Delta'(F)} p}{\prod_{p | \Delta'(F)} (p-k)}$. Using this, the identity

$$\sum_{b=0}^{M-1} \text{card}(A_b) = \text{card}(A)$$

and (2), it follows that there exists a b_0 such that

$$\text{Card}(A_{b_0}) \geq c(F) \delta \prod_{p \leq z} (p-k)^{-1} \frac{N}{\log^k N} = c(F) \delta \prod_{p \leq z} \left(1 - \frac{k}{p}\right)^{-1} \frac{N}{M \log^k N},$$

where $c(F) = c_1(F) \prod_{p | \Delta'(F)} \frac{p-k}{p}$ with $c_1(F)$ as in (2). The lemma follows using this, and Mertens formula. \square

Let $b_0 \in \{0, 1, \dots, M-1\}$ be as provided by Lemma 0.5. Without any loss of generality, we may assume that $c_1, \dots, c_r > 0$ and $c_{r+1}, \dots, c_s < 0$. Let

$$c = c_1 + \dots + c_r.$$

Let $P \in [cN/M, 2cN/M]$ be a prime and A' denote the image of A_{b_0} in $\mathbb{Z}/P\mathbb{Z}$ under the natural projection map. The set A_{b_0} contains a non-trivial solution of L if and only if A' contains a non-trivial solution of L . We shall prove Theorem 0.1 by showing that A' contains a non trivial solution.

For any set $C \subset \mathbb{Z}/P\mathbb{Z}$, we set $d(C) = \frac{\text{card}(C)}{P}$ to denote the density of C in $\mathbb{Z}/P\mathbb{Z}$. Given any set $C \subset \mathbb{Z}/P\mathbb{Z}$, let $f_C : \mathbb{Z}/P\mathbb{Z} \rightarrow \mathbb{R}_0^+$ be the function defined as $f_C(n) = \frac{1}{d(C)} I_C(n)$. For any function $f : \mathbb{Z}/P\mathbb{Z} \rightarrow \mathbb{C}$, we set $\mathbb{E}(f) := \frac{1}{P} \sum_{n \in \mathbb{Z}/P\mathbb{Z}} f(n)$. Then we may verify that for any set C , we have $\mathbb{E}(f_C) = 1$. Given any integer $l \geq 1$, we write $\|f\|_l := (\mathbb{E}(|f|^l))^{1/l}$.

The Fourier transform of f is a function $\widehat{f} : \mathbb{Z}/P\mathbb{Z} \rightarrow \mathbb{C}$ defined as $\widehat{f}(t) = \mathbb{E}(f(y) \exp(2\pi i ty))$. We also set

$$\Lambda_L(f) := \sum_{n_1, \dots, n_s \in \mathbb{Z}/P\mathbb{Z}, \sum_i c_i n_i = 0} \prod_{i=1}^s f(n_i).$$

The following identity is easy to verify:

$$\Lambda_L(f) = P^{s-1} \sum_{t \in \mathbb{Z}/P\mathbb{Z}} \prod_{i=1}^s \widehat{f}(c_i t).$$

Let G be a finite commutative group. Given functions $f, g : G \rightarrow \mathbb{C}$, we define the convolution function $f * g : G \rightarrow \mathbb{C}$ as follows:

$$f * g(n) = \frac{1}{|G|} \sum_{y \in G} f(n - y)g(y). \quad (6)$$

Proposition 0.6. *Let $A' \subset \mathbb{Z}/P\mathbb{Z}$ be as above and suppose $\delta > \log^{-100} P$. Let $B \subset [-\frac{P}{4}, \frac{P}{4}]$ with $\text{card}(B) \geq \log^{k+101} P$, then given any integer $l \geq 2$, we have*

$$\Lambda_L(f_{A'} * f_B) \geq c_1 h_L(c_2 \delta^{\frac{l}{l-1}}) P^{s-1}, \quad (7)$$

where δ is as defined in (3) and $c_1, c_2 > 0$ are constant depending only upon F, l and the linear equation L .

Proposition 0.7. *Let $\epsilon_1, \epsilon_2 > 0$ be real numbers. Let $A' \subset \mathbb{Z}/P\mathbb{Z}$ be as above and let $S_{\epsilon_1} \subset \mathbb{Z}/P\mathbb{Z}$ be the set defined as $S_{\epsilon_1} = \text{Spec}_{\epsilon_1}(f_{A'}) = \{t \in \mathbb{Z}/P\mathbb{Z} : |\widehat{f_{A'}}(t)| > \epsilon_1\}$. Let $B \subset \mathbb{Z}/P\mathbb{Z}$ such that for every $t \in S = \bigcup_i c_i c_1^{-1} \cdot S_{\epsilon_1}$, we have $|\widehat{f_B}(t) - 1| \leq \epsilon_2$, then we have*

$$|\Lambda_L(f_{A'}) - \Lambda_L(f_{A'} * f_B)| \leq c(F) \frac{\epsilon_2 + \epsilon_1^{0.5}}{\delta^6} P^{s-1},$$

where δ is as defined in (3) and $c(F) > 0$ is a constant depending only upon F .

Let $G(X) = F(b + XM)$ be the polynomial with integer coefficients and let $S \subset \mathbb{Q}$ be the set of roots of G . For proving Proposition 0.6, we shall use the following result, which we prove using beta sieve.

Proposition 0.8. *Let h_1, h_2, \dots, h_r be distinct integers with $|h_i| \leq N^{100} \forall i$. Moreover suppose for $i \neq j$, we have $h_i - h_j \notin (S - S) \cap \mathbb{Z}$, where S is the set of roots of the polynomial $G(X) = F(b + XM)$. Then we have*

$$\text{Card}((A_{b_0} + h_1) \cap \dots \cap (A_{b_0} + h_r)) \leq c(F, r) \frac{N \log^{kr} z}{M \log^{kr} N}, \quad (8)$$

where $c(F, r) > 0$ is a constant depending only upon F and r , and in particular does not depend upon h_i 's.

1 Proof of Proposition 0.6

In this section, we shall prove Proposition 0.6 using Proposition 0.8.

Given any $f : \mathbb{Z}/P\mathbb{Z} \rightarrow \mathbb{R}^+$, let $D(f)$ be the subset of $\mathbb{Z}/P\mathbb{Z}$ defined by $D(f) := \{n \in \mathbb{Z}/P\mathbb{Z} : f(n) > 1/2\}$. The following two lemmas are easy to verify.

Lemma 1.1. *Let $f : \mathbb{Z}/P\mathbb{Z} \rightarrow \mathbb{R}^+$ be a function with $\mathbb{E}(f) = 1$. Then we have $\frac{1}{P} \sum_{n \in D(f)} f(n) \geq \frac{1}{2}$.*

Proof. The result follows by observing that $\mathbb{E}(f) = \frac{1}{P} \sum_{n \notin D(f)} f(n) + \frac{1}{P} \sum_{n \in D(f)} f(n)$ and $\frac{1}{P} \sum_{n \notin D(f)} f(n) \leq \frac{1}{P} \sum_{n \in \mathbb{Z}/P\mathbb{Z}} \frac{1}{2} \leq \frac{1}{2}$. \square

Lemma 1.2. *For any $f : \mathbb{Z}/P\mathbb{Z} \rightarrow \mathbb{R}^+$ with $\text{Card}(D(f)) \geq \eta P$, we have*

$$\Lambda_L(f) \geq \frac{1}{2^s} h_L(\eta) P^{s-1}.$$

For this we need the following result which follows using the arguments from [5] and [6].

Theorem 1.3. *Let $C \subset \mathbb{Z}/P\mathbb{Z}$ be a set with the following properties. There exists a subset $S' \subset \mathbb{Z}/P\mathbb{Z}$ with $S' = -S'$, $0 \in S'$ and $\text{card}(S') \leq t$ such that given any integer $l \geq 2$ and $h_1, \dots, h_l \in \mathbb{Z}/P\mathbb{Z}$ with $h_i - h_j \notin S'$ for $i \neq j$, we have*

$$d((C + h_1) \cap \dots \cap (C + h_l)) \leq \frac{c(l)}{\beta^l} d(C)^l, \quad (9)$$

for some $\beta \leq 1$ and where $c(l) > 0$ is a constant depending only upon l . Then for any $B \subset \mathbb{Z}/P\mathbb{Z}$ with $\text{card}(B) \geq \frac{1}{d(C)}$, we have

(i) *the cardinality of the set $D := \{n \in \mathbb{Z}/P\mathbb{Z} : f_C * f_B \geq 1/2\}$ is at least $c\beta^{l/(l-1)}P$ and*

(ii) *and*

$$\Lambda_L(f_C * f_B) \geq \frac{1}{2^s} h_L \left(c\beta^{l/(l-1)} \right) P^{s-1}, \quad (10)$$

where $c > 0$ is a constant depending only upon t and l .

First we prove Proposition 0.6 using Theorem 1.3 and Proposition 0.8.

Proof of Proposition 0.6. Let $S \subset \mathbb{Q}$ be the set of roots of the polynomial $G(X) = F(b + MX) \in \mathbb{Z}[X]$ as in Proposition 0.8 and $\pi : \mathbb{Z} \rightarrow \mathbb{Z}/P\mathbb{Z}$ be the natural projection map. We shall prove the proposition by showing that the assumptions of Theorem 1.3 are satisfied with $C = A'$ and $S' = \pi((S - S) \cap \mathbb{Z})$ and $t = k^2$ and $\beta = \delta$.

Since $\text{card}(B) \geq (\log P)^{k+101}$ and $\delta \geq \frac{1}{(\log P)^{100}P}$, using Lemma 0.5, it follows that we have $\text{card}(B)d(C) \geq 1$.

We have $S' = -S'$, $0 \in S'$ and

$$\text{card}(S') \leq \text{card}(S - S) \leq \text{card}(S)^2 \leq k^2.$$

Let $h_1, \dots, h_l \in \mathbb{Z}/P\mathbb{Z}$ be such that for $i \neq j$, we have $h_i - h_j \notin S'$. The result follows by showing that (9) holds with $\beta = \delta$, where δ is as in (3). Given $x \in \mathbb{Z}/P\mathbb{Z}$, let \tilde{x} be

the integer in $[0, P)$ with $\pi(\tilde{x}) = x$. By re-ordering h'_i 's, if necessary, we may assume that $\widetilde{h_1} > \widetilde{h_2} > \dots > \widetilde{h_l}$.

Given any $n \in \cap_i (C + h_i)$, it follows that $\widetilde{n - h_i} \in A_{b_0}$ for all i . Now we observe a relation between $\widetilde{n - h_i}$ and $\widetilde{n - h_1}$. For this note that for any i , we have $\widetilde{n - h_1 + \widetilde{h_1} - \widetilde{h_i}} \in [0, 2p)$. If $\widetilde{n - h_1 + \widetilde{h_1} - \widetilde{h_i}} \in [0, p)$ then we have $\widetilde{n - h_i} = \widetilde{n - h_1 + \widetilde{h_1} - \widetilde{h_i}}$ and if $\widetilde{n - h_1 + \widetilde{h_1} - \widetilde{h_i}} \in [p, 2p)$, then we have $\widetilde{n - h_i} = \widetilde{n - h_1 + \widetilde{h_1} - \widetilde{h_i}} - P$. Using this it follows there exists $j \leq l$ such that

$$\widetilde{n - h_1} \in A^j,$$

where $A^j = \cap_{i=1}^j (A_{b_0} + \widetilde{h_i} - \widetilde{h_1}) \cap_{i=j+1}^r (A_{b_0} + \widetilde{h_i} - \widetilde{h_1} + P)$. Therefore it follows that

$$\text{Card}(\cap_i (C + h_i)) \leq \sum_{j=1}^r \text{Card}(A^j).$$

Since the condition $h_i - h_j \notin S'$ implies that for any $m \in \mathbb{Z}$, we have $\widetilde{h_i} - \widetilde{h_j} + mP \notin S - S$, using Proposition 0.8, it follows that for any j , we have $\text{Card}(A^j) \leq c(F, r) \frac{N \log^{kr} z}{M \log^{kr} N}$ and hence

$$d(\cap_i (C + h_i)) \leq c(F, r) \frac{\log^{kr} \log N}{\log^{kr} N}. \quad (11)$$

Since $\text{Card}(C) = \text{Card}(A_{b_0})$, using Lemma 0.5, we have

$$\frac{\log^k \log N}{\log^k N} \leq \frac{c(F) d(C)}{\delta}. \quad (12)$$

Therefore using (11) and (12), it follows that (9) holds with $\beta = \delta$. Hence the result follows. \square

Now we shall prove Theorem 1.3. For this we use the following observation from [6].

Proposition 1.4. *Let $f : \mathbb{Z}/P\mathbb{Z} \rightarrow \mathbb{R}^+$ be a non negative real valued function with $\mathbb{E}(f) = 1$. Then if $\|f\|_l \leq \frac{c}{\beta}$ for some integer $l \geq 2$, then we have*

$$\text{Card}(D(f)) \geq (c^{-1} \beta)^{l/(l-1)} P.$$

Proof. Using Lemma 1.1, we have

$$\frac{1}{P} \sum_{n \in D(f)} f(n) \geq 1/2.$$

Moreover we have using Hölders inequality

$$\frac{1}{P} \sum_{n \in D} f(n) \leq \|f\|_l \left(\frac{\text{card}(D)}{P} \right)^{1/q},$$

where $q > 1$ is a real number satisfying $\frac{1}{l} + \frac{1}{q} = 1$. Hence we have $\text{card}(D) \geq (c_1^{-1} \beta)^{l/(l-1)} P$ as claimed. \square

Proposition 1.5. *With the notations as in Theorem 1.3, we have*

$$\|f_C * f_B\|_l \leq \frac{c}{\beta}, \quad (13)$$

where $c > 0$ is a constant depending only upon t and l .

For proving Proposition 1.5, we first observe the following equality which is easy to verify:

$$\|f_C * f_B\|_l^l = \frac{1}{P \text{card}(B)^l d(C)^l} \sum_{y_i \in B} \text{card}((C - y_1) \cap \cdots \cap (C - y_l)). \quad (14)$$

Given $\tilde{y} = (y_1 \cdots y_l) \in B^l$, let $G(\tilde{y})$ be the graph with vertex set equal to $\{y_1 \cdots y_l\}$ and y_i is joined by an edge to y_j if and only if $y_i - y_j \in S'$, where S' is as in Theorem 1.3. Let $C(G(\tilde{y}))$ denotes the number of connected components of $G(\tilde{y})$. Given $G(\tilde{y})$ with $C(G(\tilde{y})) = r$, let $D(G(\tilde{y}))$ be a subset of $\{1, \dots, l\}$ with $\text{card}(D(G(\tilde{y}))) = r$ and for $i, j \in D(G(\tilde{y}))$ with $i \neq j$, we have y_i and y_j belongs to different connected components of $G(\tilde{y})$.

Lemma 1.6. *Let $(y_1 \cdots y_l) \in B^l$ with $C(G(y_1 \cdots y_l)) = r$. Then we have*

$$\text{Card}((C - y_1) \cap \cdots \cap (C - y_l)) \leq \frac{c(r) P d(C)^r}{\beta^r},$$

where $c(r) > 0$ is a constant depending only upon r .

Proof. We have

$$\text{Card}((C - y_1) \cap \cdots \cap (C - y_l)) \leq \text{Card}(\cap_{j \in D(G(\tilde{y}_l))} (C - y_j)).$$

We have $\text{card}(D(G(\tilde{y}_l))) = r$ and for $i, j \in D(G(\tilde{y}_l))$ with $i \neq j$, the element $y_i - y_j$ does not belong to S' . Therefore the result follows using (9). \square

The following lemma is easy to verify.

Lemma 1.7. *Let $\tilde{y} \in B^l$. If y_i and y_j belongs to the same connected components of $G(\tilde{y})$, then $y_i - y_j \in lS'$.*

Using this we prove the following lemma.

Lemma 1.8. *The number of $\tilde{y}_l \in B^l$ with $C(G(\tilde{y}_l)) = r$ is at-most $c(t, r) (\text{card}(B))^r$ where $c(t, r) > 0$ is a constant depending only upon r and t . We may take $c(t, r) = \binom{l}{r} (rt)^{2t^2}$.*

Proof. Let J be a subset of $\{1, \dots, l\}$ with $\text{card}(J) = r$. First we obtain an upper bound for the number of $\tilde{y}_l \in B^l$ such that $D(G(\tilde{y}_l)) = J$. For this we note that for any $i \in \{1, \dots, l\} \setminus J$, there exists some $j \in J$ such that $y_i - y_j \in lS'$. Hence the number of such $\tilde{y}_l \in B^l$ is at-most $(r \text{card}(lS'))^{l-r} \text{card}(B)^r$. Since there are $\binom{l}{r}$ many different sets J possible, the lemma follows. \square

Proof of Proposition 1.5. Using (14), it follows that

$$\|f_C * f_B\|_l^l = \sum_{r=1}^l \frac{1}{P \operatorname{card}(B)^l d(C)^l} \sum_{\tilde{y}_l \in B^l, C(G(\tilde{y}_l))=r} \operatorname{Card}(\cap_{i=1}^l (C - y_i)).$$

Using this and Lemmas 1.6 and 1.8, we obtain that

$$\|f_C * f_B\|_l^l \leq \sum_{r=1}^l \frac{1}{\operatorname{card}(B)^{l-r} d(C)^{l-r} \beta^r} c(r) c(t, r),$$

where $c(r)$ is as in Lemma 1.6 and $c(t, r)$ is as in Lemma 1.8. Since from assumption we have $\operatorname{card}(B)d(C) \geq 1$ and $\beta \leq 1$, the result follows with $c = l \max_r c(r) c(t, r)$. \square

The claim (i) in Theorem 1.3 is an immediate consequence of Propositions 1.4 and 1.5. The claim (ii) in Theorem 1.3 follows using this and Lemma 1.2.

2 Proof of Proposition 0.8

We shall deduce Proposition 0.8 as an easy corollary of the following result.

Theorem 2.1. *Let N' be a natural number and $G(X) = \prod_{i=1}^m (e_i X + d_i) \in \mathbb{Z}[X]$ be a polynomial with $e_i, d_i \in \mathbb{Z}$ and $|e_i| + |d_i| \leq c_1 N'^{100}$. If $\Delta(G) := \prod_i a_i \prod_{i \neq j} (e_i d_j - e_j d_i) \neq 0$, then for any $c < 1$, we have*

$$\operatorname{Card}\{n \leq N' : \gcd(G(n), P(N'^c)) = 1\} \leq c_2 \frac{N' \log^m \log N'}{\log^m N'}, \quad (15)$$

where $c_2 = c_2(m, c_1) > 0$ is a constant depending only upon m and c_1 and in particular does not depend upon N' .

Proof of Proposition 0.8. Recall that with $G(X) = F(b + MX)$, we have

$$A_b \subset \{n \leq N/M : \gcd(G(n), P(N^{1/(4k+1)})) = 1\}.$$

Using this, it follows that

$$\cap_i (A_b + h_i - h_1) \subset \{n \leq N/M : \gcd(H(n), P(N^{1/(4k+1)})) = 1\},$$

where $H(X) = \prod_{j=1}^r F'(X + h_1 - h_j)$. The assumption that $h_i - h_j \notin (S - S)$ implies that the discriminant of G is non-zero. Using Theorem 2.1 with $N' = \frac{N}{M}$ and G being the polynomial as above, we obtain that

$$\operatorname{card}(\cap_i (A_b + h_i - h_1)) \leq c_2 \frac{N \log^{kr} \log N}{M \log^{kr} N},$$

where c_2 is a constant depending only upon l and F . The result follows using this and the observation that $\operatorname{card}(\cap_i (A_b + h_i)) = \operatorname{card}(\cap_i (A_b + h_i - h_1))$. \square

Let $G \in \mathbb{Z}[X]$ be a polynomial of degree m . For any prime p , let ν_p denotes the number of $x \in \mathbb{Z}/p\mathbb{Z}$ such that $G(x) \equiv 0 \pmod{p}$. For any prime p and integer n , we set $g(p) = \frac{\nu_p}{p}$. Then it is easy to verify that for any real numbers $1 \leq w \leq z$, we have

$$\prod_{w \leq p \leq z} (1 - g(p))^{-1} \leq K \left(\frac{\log z}{\log w} \right)^m, \quad (16)$$

where K is an absolute constant. We also have

$$\sum_{n \leq x, G(n) \equiv 0 \pmod{d}} 1 = xg(d) + r(d),$$

with $|r(d)| \leq g(d)d$. Then we have the following result

Theorem 2.2. [2, Theorem 6.9, page number 69] Let $z \geq 2$ and $D \geq z^{9m+1}$. Then we have

$$\text{Card}\{n \leq x : \gcd(G(n), P(z)) = 1\} \leq (1 + K^{10} e^{9m-s}) x \prod_{p \leq z} (1 - g(p)) + \sum_{d \leq D} |r(d)|, \quad (17)$$

where $s = \log D / \log z$.

Proof of Theorem 2.1. We have

$$\sum_{d \leq D} |r(d)| \leq \sum_{d \leq D} g(d)d \leq D \prod_{p \leq D} (1 + g(p)) \ll D \log^m D.$$

Now $g(p) = \frac{m}{p}$ for all p not dividing $\Delta(G)$. From the assumption, we have $\Delta(G) = \prod_{i=1}^m e_i \prod_{i \neq j} (e_i d_j - e_j d_i) \leq c_1^{200} N'^{200m}$. Therefore the number of primes dividing $\Delta(G)$ is at-most $c(m, c_1) \log N'$, where $c(m, c_1)$ is a constant depending only upon m and c_1 . Hence

$$\prod_{p \leq z} (1 - g(p)) \leq \prod_{p \leq z} \left(1 - \frac{m}{p}\right) \prod_{p \leq c(m, c_1) \log N'} \left(1 - \frac{m}{p}\right)^{-1} \leq c(m, c_1) \frac{\log^m \log N'}{\log^m z}. \quad (18)$$

Therefore using Theorem 2.2 with $D = \frac{N'}{\log^{2m+1} N'}$ and $z = D^{\frac{1}{9m+1}}$, we obtain the result if $c < \frac{1}{9m+1}$. The result for larger c follows using this and observing that $\text{Card}\{n \leq N' : \gcd(G(n), P(z)) = 1\}$ is a decreasing function of z . \square

3 Proof of Proposition 0.7

Let $G(X) = \prod_{i=1}^m (e_i X + d_i)$ be a polynomial with $e_i, d_i \in \mathbb{Z}$ and $\Delta(G) \neq 0$. Moreover we shall assume that G is non-degenerate. The following result is a rewording of [3, Proposition 4.2].

Proposition 3.1. Let R, N be large numbers such that $1 \ll R \ll N^{1/10}$ and let G be a polynomial as above. Let $h : \mathbb{Z}/N\mathbb{Z} \rightarrow \mathbb{C}$ be a function satisfying the following:

$$h(n) \neq 0 \implies \gcd(G(n), P(R)) = 1, \quad (19)$$

where $n \in [1, N]$. Then for any real number $l > 2$, we have

$$\left(\sum_{t \in \mathbb{Z}/N\mathbb{Z}} |\hat{h}(t)|^l \right)^{2/l} \leq c(l, m) \frac{1}{\log^m R} \prod_p \left(1 - \frac{1}{p}\right)^{-m} (1 - g(p)) \frac{1}{N} \sum_n |h(n)|^2, \quad (20)$$

where $c(l, m) > 0$ is a constant depending only upon l and m .

Applying this result with $G(X) = F(b_0 + MX)$, $N = P$ and $h = f_{A'}$, we obtain

Corollary 3.2. *Given $c_1, \dots, c_s \in (\mathbb{Z}/P\mathbb{Z})^*$ and m_1, m_2, \dots, m_s with $\sum_i m_i > 2$,*

$$\sum_t \prod_{i=1}^s |\widehat{f}_{A'}(c_i t)|^{m_i} \leq \frac{c(F, \sum_i m_i)}{\delta^{\sum_i m_i}}, \quad (21)$$

where $c(F, l) > 0$ is a constant depending only on F and l .

Proof of Proposition 0.7. We have

$$|\Lambda_L(f_{A'}) - \Lambda_L(f_{A'} * B)| = P^{s-1} \left| \sum_t \prod_{i=1}^s \widehat{f}_{A'}(c_i c_1^{-1} t) \left(1 - \prod_{i=1}^s \widehat{f}_B(c_i c_1^{-1} t) \right) \right|. \quad (22)$$

Since for $t \in \bigcup_i c_i c_1^{-1} \cdot S_{\epsilon_1}$, we have $|\widehat{f}_B(t) - 1| \leq \epsilon_2$, it follows that for $t \in S_{\epsilon_1}$, we have $|1 - \prod_i \widehat{f}_B(c_i c_1^{-1} t)| \ll \epsilon_2$. Hence using this and (21) we have

$$\sum_{t \in S_{\epsilon_1}} \left| \prod_{i=1}^s \widehat{f}_{A'}(c_i c_1^{-1} t) \left(1 - \prod_{i=1}^s \widehat{f}_B(c_i c_1^{-1} t) \right) \right| \ll \epsilon_2 \sum_t \left| \prod_{i=1}^s \widehat{f}_{A'}(c_i c_1^{-1} t) \right| \leq \epsilon_2 \frac{c(F, L)}{\delta^s}. \quad (23)$$

For $t \notin LS_{\epsilon_1}$, we have $|f_{A'}(t)|^{1/2} \leq \epsilon_1/2$. Therefore the contribution in right hand side of (22) coming from such t is at-most and hence we using (21), we have

$$\epsilon_1^{1/2} \sum_{t \notin S_{\epsilon_1}} \left| \widehat{f}(t)^{1/2} \prod_{i=2}^s \widehat{f}_{A'}(c_i c_1^{-1} t) \right| \ll \epsilon_1^{1/2} \frac{c(F, L)}{\delta^s}. \quad (24)$$

Using (22), (23) and (24), the result follows. \square

4 Relation between g_L and h_L

When the number of variables s in a translation invariant linear equation L is 3, a relation between g_L and h_L follows from the following result.

Theorem 4.1 (Varnavides theorem). *Let L be a translation invariant linear equation and g_L, g_L^* are functions as defined in Definition 0.3. Let $\eta > 0$ and $D \subset \mathbb{Z}/P\mathbb{Z}$ with $\text{card}(D) \geq \eta P$. Then the number of solution of L in D is at least*

$$\frac{\eta}{2} \frac{P(P-1)}{(g^{*-1}(\eta/2))^2}.$$

Proof. For the brevity of notation, we write t to denote $g^{*-1}(\eta/2)$. Since the assumption implies that D is non empty and hence contains at least one trivial solution of L , the result is true if $t \geq P$. Hence we may assume that $t < P$.

Given any $a \in \mathbb{Z}/P\mathbb{Z}$ and $d \in \mathbb{Z}/P\mathbb{Z} \setminus \{0\}$, let $I_{a,d} := \{a + d, \dots, a + td\}$ be an arithmetic progression of length t . We say that $I_{a,d}$ is a “good” progression, if $\text{card}(D \cap I_{a,d}) \geq \frac{\eta}{2}t$. We claim that if $I_{a,d}$ is good then $D' = D \cap I_{a,d}$ contains a non-trivial three term arithmetic progression. For this we first notice that since P is prime and d is a non zero element

of $\mathbb{Z}/P\mathbb{Z}$, we have $\text{card}(D') = \text{card}(\frac{D'-a}{d})$. Hence $\frac{D'-a}{d} \subset [1, t]$ and contains at least $\frac{\eta}{2}t$ elements. Therefore using the properties of g_L and definition of $g_L^{*-1}(\eta/2)$, it follows that $\frac{D'-a}{d}$ contains a non-trivial solution of L , which proves the claim. Now we shall obtain a lower bound for the number of good $I_{a,d}$.

Now for any fixed d_0 , we have the following identity:

$$\sum_{a \in \mathbb{Z}/P\mathbb{Z}} \text{card}(D \cap I_{a,d_0}) = t \text{card}(D).$$

This follows by observing that any $c \in D$ belongs to exactly t many I_{a,d_0} . From the above identity it follows that for any fixed d_0 , the number of good I_{a,d_0} is at least $\frac{\text{card}(D)}{2}$ which by assumption is at least $\frac{\eta}{2}P$. Now varying d_0 , we obtain that the number of good $I_{a,d}$ is at least $\frac{\eta}{2}P(P-1)$. The lemma follows using this and the observation that a given non-trivial solution of L can belong to at most t^2 many good $I_{a,d}$. \square

Using Theorem 4.1, we immediately obtain the following result.

Corollary 4.2. *Let L be a translation invariant equation in s many variables and g_L be a function as satisfying the properties as in Definition 0.3. When $s = 3$, then $h_L(\eta) = \frac{\eta}{(2g_L^{*-1}(\eta/2))^2}$ is a function satisfying the properties as in Definition 0.3 (i).*

As remarked earlier, Thomas Bloom [1] showed that there exists an absolute constant $c > 0$ depending only on L such that the function $g_L(N) = c \left(\frac{\log^5 \log N}{\log N} \right)^{s-2}$ satisfies the above properties. In this case $g_L^{*-1}(\eta) \leq \exp(c_1 \eta^{-1/(s-2)} \log^6 \log(\frac{1}{\eta}))$ with $c_1 > 0$ being a constant depending only upon L . Therefore when $s = 3$, there exists an absolute constant $c > 0$ such that we may take

$$h_L(\eta) = \exp \left(-c\eta^{-1} \log^6 \log \frac{1}{\eta} \right). \quad (25)$$

5 Proof of Theorem 0.1

Let S be as in Proposition 0.7 and $B \subset \mathbb{Z}/P\mathbb{Z}$ defined as

$$B = \text{Bohr}(S, \epsilon_2) := \{x \in \mathbb{Z}/P\mathbb{Z} : \left| \exp \left(\frac{2\pi i x t}{P} \right) - 1 \right| \leq \epsilon_2 \forall t \in S\}.$$

Then B satisfies the assumptions in Proposition 0.7. We shall choose ϵ_1 and ϵ_2 in such a way that B also satisfies the assumptions in Proposition 0.6.

Lemma 5.1 (Lemma 4.20 [7]). *Given any set $C \subset \mathbb{Z}/P\mathbb{Z}$ and any real number $\epsilon > 0$, we have*

$$\text{card}(\text{Bohr}(C, \epsilon)) \geq (\epsilon)^{|C|} P.$$

Moreover an immediate consequence of (21) is the following upper bound for the cardinality of S :

$$\text{Card}(S) \leq \frac{\epsilon_1^{-3} c(F, L)}{\delta^3}.$$

Therefore we have $\text{card}(B) \geq \log^{k+101} P$ and hence B satisfies the assumption of Proposition 0.6 provided, we have

$$\frac{\epsilon_1^{-3} c(F, L)}{\delta^3} \log(\epsilon_2) \geq -\frac{\log P}{2} \quad (26)$$

and P is sufficiently large. Therefore if (26) is satisfied, then using Propositions 0.6 and 0.7, we have

$$\Lambda_L(f_{A'}) \geq c_1 h_L(c_2 \delta^{l/(l-1)}) P^{s-1} - c(F, L) \frac{\epsilon_2 + \epsilon_1^{0.5}}{\delta^s} P^{s-1}.$$

Therefore choosing

$$\epsilon_2 = \epsilon_1^{0.5} = \frac{\delta^s c_1 h_L(c_2 \delta^{l/(l-1)})}{c(F, L)}, \quad (27)$$

we obtain

$$\Lambda_L(f_{A'}) \geq c_1 h_L(c_2 \delta^{l/(l-1)}) P^{s-1}, \quad (28)$$

where c_1 and c_2 are constants depending only upon F and the linear equation L , provided our choice of ϵ_1 and ϵ_2 satisfies (26). Since $s = 3$, with the choice of h_L provided by (25), we have that for some $c_1, c_2 > 0$, we have

$$\epsilon_1 = \exp\left(-c_1 \delta^{-l/(l-1)} \log^6 \log \frac{1}{\delta}\right), \quad \epsilon_2 = \exp\left(-c_1 \delta^{-l/(l-1)} \log^6 \log \frac{1}{\delta}\right).$$

Therefore (26) holds using the assumed lower bound for δ , provided l is chosen sufficiently large depending on ϵ , where ϵ is as in Theorem 0.1.

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